

# Self-Similar Jordan Arcs Which Do Not Satisfy OSC.

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The problem of finding explicit geometrical criteria for a system  $\mathcal{S}$  of contraction similarities, implying the open set condition, is discussed since 80-ies and still remains open.

One of such criteria is the *finite intersection property*:

The system  $\mathcal{S} = \{S_1, \dots, S_m\}$  with the attractor  $K$ , has f.i. property if for any  $i \neq j$  the set  $S_i(K) \cap S_j(K)$  is finite.

It was proved in 2007 by C.Bandt and H.Rao [2] that if a system  $\mathcal{S} = \{S_1, \dots, S_m\}$  of contraction similarities in  $\mathbb{R}^2$  with a connected attractor  $K$  has the finite intersection property, then it satisfies OSC. The authors wrote they believe that in  $\mathbb{R}^3$  this is not so.

In this paper we prove the following

**Theorem 1.** *There is such system  $\mathcal{S} = \{S_1, \dots, S_m\}$  of contraction similarities in  $\mathbb{R}^3$ , which:*

- (1) *does not satisfy OSC,*
- (2) *satisfies one-point intersection property and*
- (3) *whose attractor is a Jordan arc  $\gamma \subset \mathbb{R}^3$ .*

Notice that the statements (1) and (2) imply that the system  $\mathcal{S}$  does not satisfy weak separation property WSP.

To show the existence of such self-similar arcs we use the zipper construction [1] and prove the following three statements, which are the foundation of our approach to construction of non-WSP curves:

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First is a **general position theorem** for fractal curves, which gives the condition specifying how to get rid of intersection by small deformations of pairs of fractal curves within a given family of such pairs  $\{(\varphi(x, t), \psi(x, t))\}$ , depending on some parameter  $x \in \mathbb{R}^3$ .

**Theorem 2.** *Let  $\varphi(x, t), \psi(x, t) : B^3 \times I \rightarrow \mathbb{R}^3$  be continuous maps which*  
*(1) are  $\alpha$ -Hölder with respect to  $t$  and*  
*(2) satisfy the condition: for any  $t, s \in I$  the function  $f(x, t, s) = \varphi(x, t) - \psi(x, s)$  is bi-Lipschitz with respect to  $x$ .*  
*Then Hausdorff dimension of the set  $\Delta = \{x \in B^3 | \varphi(x, I) \cap \psi(x, I) \neq \emptyset\}$  does not exceed  $2/\alpha$ .*

Second is a **corollary of Barnsley's Collage Theorem**, that gives the conditions under which a deformation of a fractal is bi-Lipschitz on its certain subpieces.

**Proposition 3.** *Let  $\mathcal{S} = \{S_1, \dots, S_m\}, \mathcal{T} = \{T_1, \dots, T_m\}$  be systems of contractions in a complete metric space  $X$ , and  $q = \max(\text{Lip } S_i, \text{Lip } T_i)$ . Let  $\varphi : I^\infty \rightarrow K, \psi : I^\infty \rightarrow L$  be index maps for the attractors  $K(\mathcal{S})$  and  $L(\mathcal{T})$ . Let  $V$  be such bounded set, that for any  $i = 1, \dots, m$ ,  $S_i(V) \subset V$  and  $T_i(V) \subset V$ . Put  $\Delta_i(x) = T_i(x) - S_i(x)$  and suppose  $\|\Delta_i(x)\| \leq \delta$  for any  $i$  and any  $x \in V$ . Then,*

**B1)** *If for some multiindex  $\mathbf{j}$ ,  $\delta_1 < \|\Delta_{\mathbf{j}}(x)\| < \delta_2$  for any  $x \in V$  and  $\delta_1 > \frac{q_{\mathbf{j}}\delta}{1-q}$ , then for  $\sigma \in \tau_{\mathbf{j}}I^\infty$ ,*

$$\delta_1 - \frac{q_{\mathbf{j}}\delta}{1-q} \leq \|\psi(\sigma) - \varphi(\sigma)\| \leq \delta_2 + \frac{q_{\mathbf{j}}\delta}{1-q}$$

**B2)** *If for some multiindices  $\mathbf{i}, \mathbf{j}$ ,  $\delta_1 < |\Delta_{\mathbf{i}}(x) - \Delta_{\mathbf{j}}(y)| < \delta_2$  for any  $x, y \in V$  and  $\delta_1 > \frac{(q_{\mathbf{i}} + q_{\mathbf{j}})\delta}{1-q}$ , then for  $\tau \in \tau_{\mathbf{j}}I^\infty, \sigma \in \sigma_{\mathbf{i}}I^\infty$ ,*

$$\delta_1 - \frac{(q_{\mathbf{i}} + q_{\mathbf{j}})\delta}{1-q} \leq \|\psi(\sigma) - \varphi(\sigma) - \psi(\tau) + \varphi(\tau)\| \leq \delta_2 + \frac{(q_{\mathbf{i}} + q_{\mathbf{j}})\delta}{1-q}$$

And the third, most delicate, statement allows to make deformations of system  $\mathcal{S}$  and the curve  $\gamma$ , under which the violation of WSP by the system  $\mathcal{S}$  is preserved.

**Theorem 4.** 1) If two-generator subgroup  $G = \langle \xi, \eta, \cdot \rangle$ ,  $\xi = re^{i\alpha}$ ,  $\eta = Re^{i\beta}$  in  $\mathbb{C} \setminus \{0\}$  is a dense subgroup of second type, then for any  $z_1, z_2 \in \mathbb{C} \setminus \{0\}$  there is such sequence  $\{(n_k, m_k)\}$  that  $\lim_{k \rightarrow \infty} \frac{z_1 \xi^{n_k}}{z_2 \eta^{m_k}} = 1$ ,  $\lim_{k \rightarrow \infty} e^{in_k \alpha} = e^{-i \arg(z_1)}$ ,  $\lim_{k \rightarrow \infty} e^{in_k \beta} = e^{-i \arg(z_2)}$ .  
2) The set  $\{(\xi, \eta)\}$  of all pairs of generators of dense subgroups of the second type, is dense in  $\mathbb{C}^2$ .

The subgroups mentioned in the theorem were defined in [4] and we give a short summary of the results of this work in the Addendum.

## 1 Brief outline of the method.

The idea of the example is quite simple and is based on zipper construction introduced by Vladislav Aseev in [1].

Namely, a system  $\mathcal{S} = \{S_1, \dots, S_m\}$  of contractions of a metric space  $X$  is called a *zipper* with vertices  $\{z_0, \dots, z_m\}$  and signature  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ ,  $\varepsilon_i \in \{0, 1\}$ , if for any  $i = 1, \dots, m$ ,  $S_i(z_0) = z_{i-1+\varepsilon_i}$  and  $S_i(z_m) = z_{i-\varepsilon_i}$ .

We denote the attractor of a zipper  $\mathcal{S}$  by  $\gamma(\mathcal{S})$ , or simply by  $\gamma$ .

Note, that for any zipper  $\mathcal{S}$  and any linear zipper  $\mathcal{T}$  on  $[0, 1]$  having the same signature  $\varepsilon$  there is unique a Hölder continuous  $(\mathcal{S}, \mathcal{T})$ -equivariant map  $\varphi_{\mathcal{S}\mathcal{T}} : [0, 1] \rightarrow \gamma$  which is called a *linear parametrization* of  $\mathcal{S}$ . [1]

We begin with a self-similar zipper  $\mathcal{S} = \{S_1, \dots, S_{2m}\}$  in  $\mathbb{R}^3$ , whose vertices  $z_0, \dots, z_{2m}$  and similarities  $S_i$  are chosen in such way, that for some closed bicone  $V$  with vertices  $z_0, z_{2m}$ ,

(A1) for any  $S_i \in \mathcal{S}$ ,  $S_i(V) \subset V$  ;

(A2) for any such  $i, j$ , that  $|j - i| > 1$ ,  $S_i(V) \cap S_j(V) = \emptyset$

(A3) for any  $i \neq m + 1$  and  $1 \leq i \leq 2m$ ,  $S_{i-1}(V) \cap S_i(V) = \{z_i\}$ ;

At the same time, inside  $S_m(V) \cup S_{m+1}(V)$  we provide that

(A4) There are such subarcs  $\gamma_A \supset \gamma_{m-3}$ ,  $\gamma_B \supset \gamma_{m+4}$  and such sequences  $\{i_k\}$ ,  $\{j_k\}$ , that the set  $S_m(\gamma) \cap S_{m+1}(\gamma) \setminus \{z_m\}$  is a disjoint union of sets

$$S_{m+1}S_1^{i_k}(\gamma_A) \cap S_mS_{2m}^{j_k}(\gamma_B)$$

**(A5)** The sequence of pairs of maps  $S'_k = S_{m+1}S_1^{i_k}, S''_k = S_mS_{2m}^{j_k}$ , contains such subsequence  $\{S'_{k_n}, S''_{k_n}\}$ , that

$$\lim_{n \rightarrow \infty} (S'_{k_n} S_{m-3})^{-1} (S''_{k_n} S_{m+4}) = \text{Id}.$$

**(A6)** There is such linear zipper  $\mathcal{T}$  in  $[0, 1]$  that the Hölder exponent of the linear parametrization  $\varphi_{s\mathcal{T}}$  is greater than  $3/4$ .

The condition (A5) means that the system  $\mathcal{S}$  does not satisfy WSP, but (A1) – (A3) do not imply in general, that  $\gamma$  is a Jordan arc.

So, the main difficulty is to change the system  $\mathcal{S}$  slightly in such way that  $\gamma$  gets rid of all its self-intersections without violating (A4, A5).

For that reason, instead of a single zipper  $\mathcal{S}$ , in Section 2 we construct a family of self-similar zippers  $\mathcal{S}_\xi = \{S_1, \dots, S_{2m}\}$ , depending continuously on a parameter  $\xi \in D$ , where  $D$  is some domain in  $\mathbb{R}^3$ , so that for any  $\xi \in D$ ,  $\mathcal{S}_\xi$  satisfies (A1) - (A6) with the same domain  $V$ , sequences  $S'_k, S''_k$  and the same linear zipper  $\mathcal{T}$  for all  $\xi \in D$ .

The most delicate problem here is to choose such parameters for the family  $\{\mathcal{S}_\xi\}$ , that the condition (A4) would hold for all  $\mathcal{S}_\xi$ . This is guaranteed by our Theorem 4 on the properties of dense subgroups in  $\mathbb{C}^*$ , proved in [4].

After that we show that the set of the parameters  $\xi \in D$ , for which  $\mathcal{S}_\xi$  defines a Jordan arc, is dense in  $D$ . To obtain this, we apply the general position Theorem 2 the following way:

We take the family  $\{\varphi^\xi, \xi \in D\}$  of linear parametrisations  $\varphi^\xi(t) : [0, 1] \rightarrow \gamma^\xi$  of zippers  $\mathcal{S}_\xi$  by the zipper  $\mathcal{T} = \{T_1, \dots, T_m\}$ . Denoting by  $\varphi_k(\xi, t), \psi_k(\xi, t)$  the parametrisations of the subarcs  $S'_k(\gamma_A), S''_k(\gamma_B)$ , obtained by restriction of  $\varphi^\xi(t)$  to the subintervals  $T_{m+1}T_1^{i_k}(I_A), T_mT_{2m}^{j_k}(I_B)$  of  $I = [0, 1]$ , we consider the functions  $f_k(\xi, t, s) = \varphi_k(\xi, t) - \psi_k(\xi, s)$ . Applying Proposition 3, we show that the function  $f_k(\xi, t, s) = \varphi_k(\xi, t) - \psi_k(\xi, s)$ , is bi-Lipschitz with respect to  $\xi$  for each fixed  $t, s$ . By our construction,  $f_k(\xi, t, s)$  is  $\alpha$ -Hölder with respect to  $t$  and  $s$  and  $\alpha > 2/3$ .

Therefore, applying Theorem 2 to each pair of subarcs from the sequence  $S'_k(\gamma_A), S''_k(\gamma_B)$  and using Baire category argument, we get that each ball in the domain  $D$  contains such  $\xi$ , that all the intersections  $S'_k(\gamma_A) \cap S''_k(\gamma_B)$  are empty and therefore  $\gamma^\xi$  is a Jordan arc. So the set of parameters  $\xi$ , for which the system  $\mathcal{S}(\xi)$  has a Jordan attractor  $\gamma$  is dense in  $D$ .

## 2 Setting the parameters of zippers $\mathcal{S}^\xi$ .

In this section we define the parameter domain  $D$ , the zippers  $\mathcal{S}^\xi$  for any  $\xi \in D$  and the bicones  $V_i$ .

### 2.1 Polygonal lines defining the zippers $\mathcal{S}^\xi$ .

First, we define three angles  $\beta_1 < \beta_0 < \beta_2$ , necessary for our construction. Take  $\beta_0 = \arctan(1/2)$ ; let  $\mu = 0.0100512$  and put  $\beta_1 = \beta_0 - 2\mu$ ,  $\beta_2 = \beta_0 + \mu$ .

We define the domain  $D$  in  $\mathbb{R}^3$  by the equation

$$D = (1/1.02, 1.02) \times (\beta_0 - \mu, \beta_0 - \mu/2) \times (-\mu, \mu)$$

Denote a point in  $D$  by  $\xi = (\rho, \theta, \varphi)$ .

Now we define a family of zippers  $\mathcal{S}_\xi$  depending on a parameter  $\xi \in D$

So for each  $\xi \in D$  we define a polygonal line in the plane XY with vertices  $z_0, \dots, z_{2m}$ . All its vertices, except  $z_{m+1}$ , do not depend on  $\xi$ .

The similarities  $S_i$  in each zipper  $\mathcal{S}^\xi$  are the compositions of XY-plane preserving similarities sending  $\{z_0, z_{2m}\}$  to  $\{z_{i-1}, z_i\}$  and rotations in some angle  $\alpha_i$  around the axis  $\{z_{i-1}, z_i\}$ .

In this family, only the point  $z_{m+1}$  and the maps  $S_{m+1}, S_{m+2}$  and  $S_{m+4}$  depend on  $\xi$ , others being the same for any  $\xi \in D$ .

Mentioning the points, maps or subset of the attractor of each system  $\mathcal{S}_\xi$  we will write  $z_i^\xi$  or  $S_i^\xi$  only in the cases when it is needed for our argument, otherwise we will not mention the parameter  $\xi$ , assuming the dependence by default.

The table below shows the x and y coordinates of the vertices  $z_i$ , the contraction ratios  $q_i = \frac{\|z_i - z_{i-1}\|}{\|z_{2m} - z_0\|}$  and rotation angles  $\alpha_i$ .

No	0	1	2	.....	$m-5$	$m-4$	$m-3$	$m-2$	$m-1$	$m$
$x :$	-3	$6q_1 - 3$	-1	...	-1	-1	-0.92	-0.899	-0.447	0
$y :$	0.8	0.8	0.8	...	1.750	1.8	1.84	1.798	0.894	0
$q_i :$	-	$q_1$	$\frac{1}{3} - q_1$	...	...	0.008	0.015	0.008	0.168	$\frac{1}{6}$
$\alpha_i :$	-	$\alpha_1$	0	...	...	0	0	0	0	0



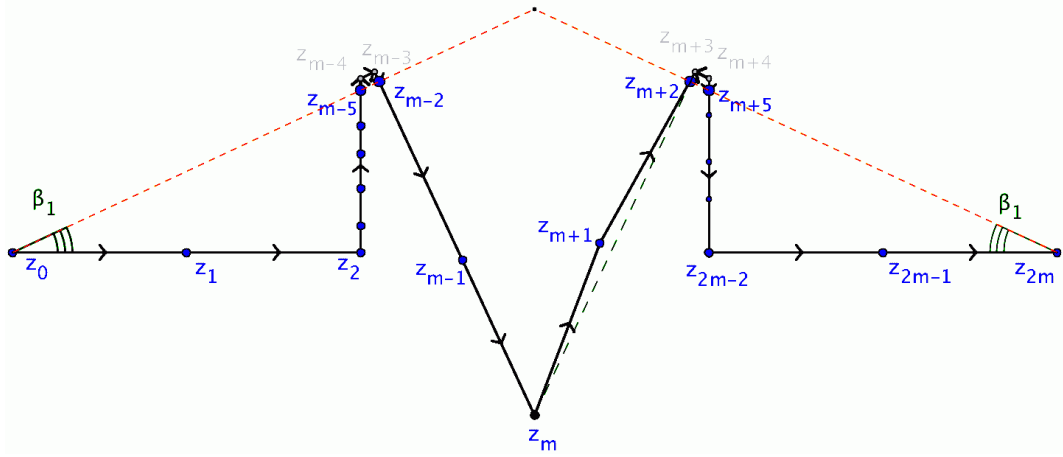


Figure 1:

6.  **$\beta_1$  triangle.** At the same time the points  $z_{m-5}, z_{m-2}, z_{m+2}, z_{m+5}$  lie on the sides of an isosceles triangle with angles  $\beta_1$  at its base  $z_0 z_{2m}$ .

7. **Points near  $z_m$ .** The points  $z_{m-3}, z_{m-2}, z_{m-1}, z_m$  lie on a line, forming the angle  $\beta_0$  with the vertical axis and  $\|z_m - z_{m-1}\| = 1$ . The same is true for their symmetrical counterparts  $z_{m+3}, z_{m+2}, z_m$ . The point  $z_{m+1}$  is slightly shifted to the left and upwards.

8. **Omitted entries.** The points  $z_3, \dots, z_{m-6}$  and  $z_{m+6}, \dots, z_{2m-3}$  divide the intervals  $(z_2, z_{m-5})$  and  $(z_{m+5}, z_{2m-2})$  into sufficiently small equal parts, therefore their coordinates and corresponding  $q_i$  need not be mentioned.

## 2.2 The similarities $S_i$

Each zipper  $\mathcal{S}^\xi = \{S_1, \dots, S_{2m}\}$  has vertices  $(z_0, \dots, z_{2m})$  and signature  $(0, 0, \dots, 0, 1, 0, \dots, 0)$ , where only  $\varepsilon_{m+4} = 1$ . This means that  $S_{m+4}$  reverses the order, i.e.  $S_{m+4}(z_0) = z_{m+4}$  and  $S_{m+4}(z_{2m}) = z_{m+3}$ . See Fig. 6.

$S_1$  and  $S_{2m}$  are defined as the compositions of contractions with ratios  $q_1, q_{2m}$  with fixed points  $z_0$  and  $z_{2m}$  respectively and rotations in angles  $\alpha_1$  and  $-\alpha_{2m}$  around the real axis.

The map  $S_{m+1}^\xi$  is a composition of a plane similarity sending  $z_0$  to  $z_m$  and  $z_{2m}$  to  $z_{m+1}$  and a rotation around the line  $z_m z_{m+1}$  in the angle  $\varphi$ , which is the third coordinate of the parameter  $\xi$ .

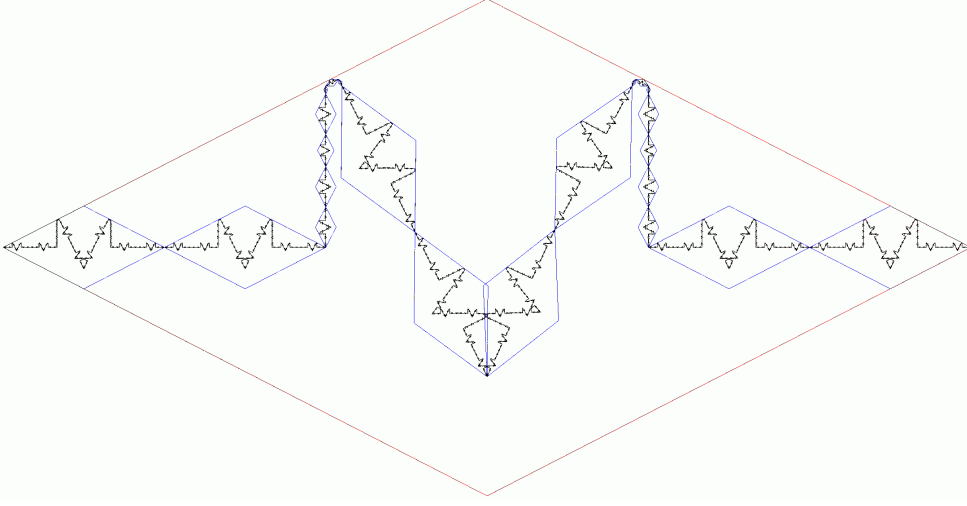


Figure 2: The attractor  $\gamma$  of al zipper  $\mathcal{S}^\xi$  and the sets  $V_i$ .

The map  $S_{m+2}$  preserves the plane  $XY$ , but depends on  $\xi$  because its value at  $z_0$  is  $z_{m+1}^\xi$ .

Finally, the map  $S_{m+4}$  is a composition of a (fixed) similarity map of a plane  $XY$  sending  $z_0$  to  $z_{m+4}$  and  $z_{2m}$  to  $z_{m+3}$  and a rotation in an angle  $\alpha_{m+4}$  around the line  $z_{m+3}z_{m+4}$ . The angle  $\alpha_{m+4}$  depends on the coordinate  $\theta$  of  $\xi$  and will be defined in the next section.

All the other maps  $S_2, \dots, S_{2m-1}$  are the similarities, preserving  $XY$  plane.

The similarity ratios  $q_i$  of  $S_i$  are equal to  $|z_i - z_{i-1}|/6$ . Direct computation shows that when  $m \geq 12$  and  $q_3 = \dots = q_{m-5}$ , the similarity dimension of  $\mathcal{S}$  is less than 1.28 for any  $\xi \in D$ .

### 2.3 Bicones and the sets $A$ and $B$ .

Let  $V'$  be a rhombus whose diagonal is  $z_0, z_{2m}$  and the angle between diagonal and its sides is  $\beta_2$ .

The value of  $\beta_2$  was chosen as minimal of those ones, for which two small copies of  $V'$  with diagonals  $z_{m-3}z_{m-4}$  and  $z_{m+4}z_{m+3}$  respectively lie inside the large  $V'$ .

For each other  $i$ , the copy of  $V'$  whose diagonal is  $z_{i-1}z_i$  also lies inside

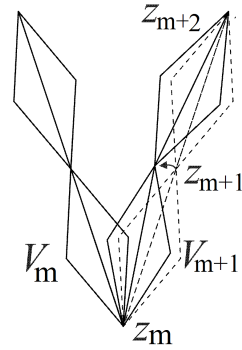


Figure 3:



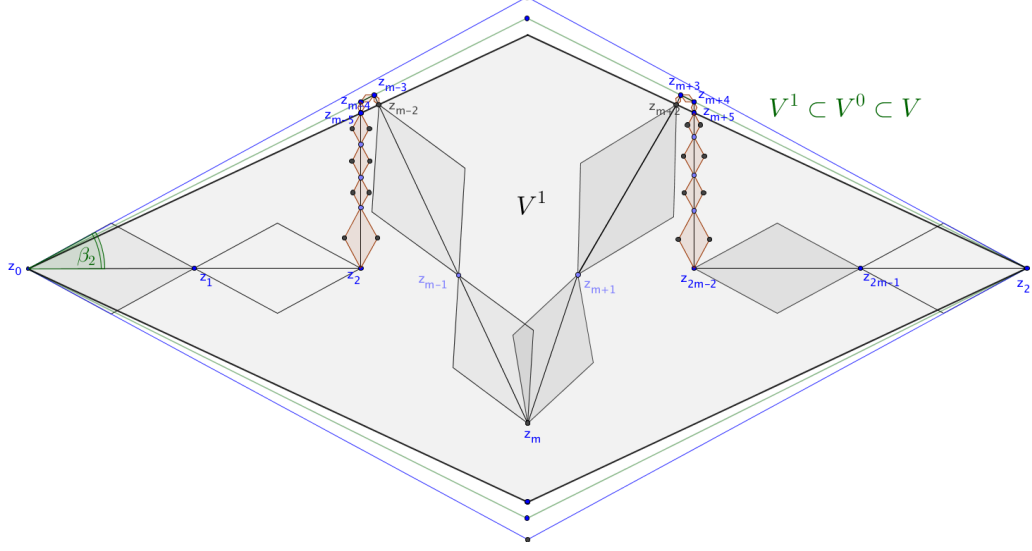


Figure 4: The bicones  $V^1 \subset V^0 \subset V$ :  $\beta_2$  is enlarged to make the inclusion visible.

$V'$ .

Getting to the space, we replace  $V'$  by a bicone  $V$  with the same axis  $z_0, z_{2m}$  and angle  $\beta_2$  between the axis and a generator.

We denote by  $V_i$  the images of the bicone  $V$  under similarities sending  $z_0, z_{2m}$  to  $z_{i-1}, z_i$ . They satisfy the relations **(A1)–(A3)** from Section 1.

These properties are valid for any choice of  $\xi \in D$ . (See Fig.3)

Along with the bicones  $V_i$ , we will consider the bicones  $V_i^0$  with the same vertices and with the angle  $\beta_0$  between axis and generator and the bicones  $V_i^1$  with the same vertices and with the angle  $\beta_1$  between axis and generator.

**Lemma 5.** *For any  $\xi \in D$ ,*

- (i) *for any  $i = 1, \dots, 2m$ ,  $V_{i-1}^0 \cap V_i^0 = \{z_i\}$ ;*
- (ii)  *$V_m \cap V_{m+1}^1 = V_m^1 \cap V_{m+1} = \{z_m\}$ ;*
- (iii) *the dihedral angle with edge  $z_m z_{m+1}$  containing common generators of  $V_m$  and  $V_{m+1}$ , is no greater than 0.545*
- (iv)  *$V_m^0 \cap V_{m+1}^0 \neq \emptyset$  and the dihedral angle with the edge  $z_m z_{m+1}$  and sides, containing common generators of  $V_m^0$  and  $V_{m+1}^0$ , lies between 0.224 and 0.317.*

**Proof** Since the angle between the axes is greater than  $2\beta - \mu$ , we have (i) and (ii).

The upper bound of such angle for  $V_m \cap V_{m+1}$  is  $2 \arccos \frac{\tan(\beta - \mu/2)}{\tan(\beta + \mu)} = 0.545$  gives us (iii). Now, the upper and lower bounds for the angle for  $V_m^0 \cap V_{m+1}^0$  are  $2 \arccos \frac{\tan(\beta - \mu/2)}{\tan(\beta)}$  and  $2 \arccos \frac{\tan(\beta - \mu/4)}{\tan(\beta)}$ , which gives (iv). ■

## 2.4 The value of $\alpha_{m+4}$ .

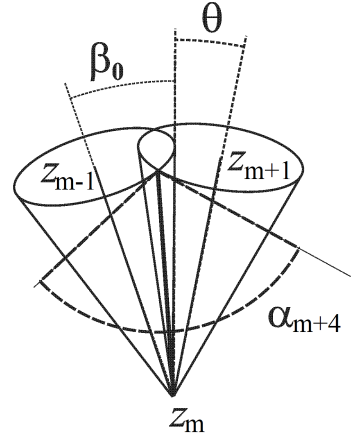
Consider the bicones  $V_m^0$  and  $V_{m+1}^0$ . They have common vertex  $z_m = 0$ . The bicone  $V_m^0$  stays fixed, while  $V_{m+1}^0$  changes its position depending on its second variable vertex  $z_{m+1}^\xi$ , and the angle between the axis of  $V_{m+1}$  and OY is  $\theta \in (\beta - \mu, \beta - \mu/2)$ , therefore  $V_m^0$  and  $V_{m+1}^0$  have nonempty intersection.

The angle  $\alpha_{m+4}$  is equal to an angle between the tangent planes to the surfaces of  $V_m^0$  and  $V_{m+1}^0$  at points of their common generator. By spherical cosine theorem, it is equal to

$$\alpha_{m+4}(\theta) = \arccos(4 - 5 \cos(\beta_0 + \theta)).$$

The values of the derivative  $\cos(\alpha_{m+4}(\theta))'$  for  $\theta \in [\beta - \mu, \beta - \mu/2]$  lie in the interval  $(-20, -14)$ . Therefore, for any  $\theta_1, \theta_2 \in (\beta - \mu, \beta - \mu/2)$ ,

$$|\alpha_{m+4}(\theta_1) - \alpha_{m+4}(\theta_2)| < 20|\Delta\theta|$$



## 2.5 The sets A and B.

There are 2 sets, formed from  $V_i$ -s, which will be needed for our considerations: the set  $A = V_{m-4} \cup V_{m-3} \cup V_{m-2}$  and the set  $B = V_{m+3} \cup V_{m+4} \cup V_{m+5}$ :

The sets  $A$  and  $B$  lie outside the bicone  $V^1$ , but intersect  $V^0$  so that the axes of  $V_{m-3}$  and  $V_{m+4}$  are contained in the boundary of  $V^0$  .

### Some inequalities for the sets A and B.

For any two points  $x$  and  $y$  in the set  $A$  (or both in the set  $B$ ), we have  $\frac{\|x - z_0\|}{\|y - z_0\|} < 1.06$  and  $\frac{\|x - Pr_1(x)\|}{\|y - Pr_1(y)\|} < 1.095$ . At the same time, the dihedral

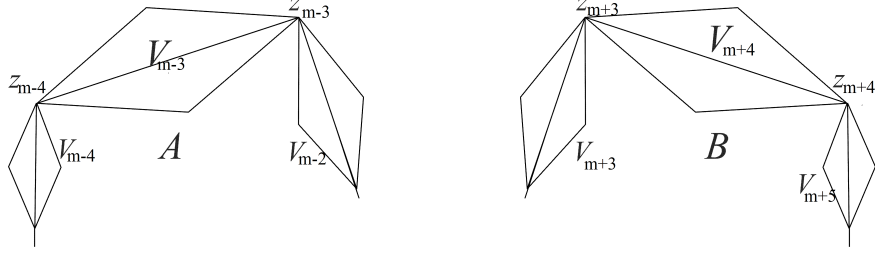


Figure 5: The sets A and B.

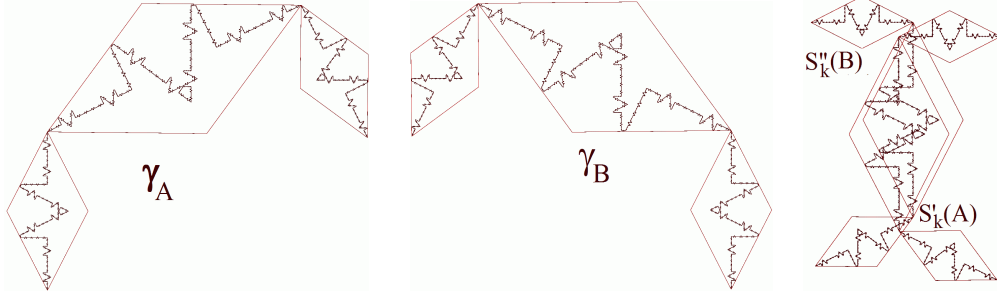


Figure 6: The sets  $\gamma_A = \gamma \cap A$  and  $\gamma_B = \gamma \cap B$ . The position of  $\gamma_{m+4}$  is due to signature  $\varepsilon_{m+4} = 1$ . So their images  $S'_k(\gamma_A)$  and  $S'_k(\gamma_B)$  are parallel.

angle, containing A and B, whose edge is the line  $z_0 z_{2m}$ , is no greater than  $2\sqrt{5}\mu$ .

Therefore, the set A lies in a domain defined by the inequalities

$$R \leq |x - z_0| \leq 1.06R; \quad -\sqrt{5}\mu \leq \varphi \leq \sqrt{5}\mu; \quad \beta_0 - 2\mu \leq \theta \leq \beta_0 + \mu,$$

where  $\varphi$  and  $\theta$  are the azimuth and polar angles for the point  $x$  in spherical coordinates with the origin  $z_0$ , real line being the polar axis and the azimuth direction being parallel to  $OY$ .

Here  $R = 2.214$  is the distance from  $z_0$  to the nearest point of A.

Direct computation shows that

**Lemma 6.** *The set A can be covered by a ball W of radius  $0.036R$  with the center defined by*

$$|x - z_0| = 1.03R; \quad \varphi = 0; \quad \theta = \beta_0 - \mu/2 \quad \blacksquare$$

By symmetry, analogous inequalities hold for the points in the set  $B$ .

### 3 Checking the properties of the family of zippers $\mathcal{S}_\xi$ .

Now we can verify the properties **A1** – **A6** for any  $\xi \in D$  and evaluate the difference  $S_i^\xi(x) - S_i^\eta(x)$  for  $x \in V$ . Fortunately **A1**–**A3** are obvious so we proceed to **A6**, **A5**, **A4**.

#### 3.1 Defining linear zipper $\mathcal{T}$ and checking **A6**.

As it was proved in ([1], Lemma 1.1), for any linear zipper  $\mathcal{T} = \{T_1, \dots, T_{2m}\}$  with attractor  $K(\mathcal{T}) = [0, 1]$  and with the same signature  $\varepsilon$  as the zipper  $\mathcal{S}$ , there is unique continuous function  $\varphi_{\mathcal{S}\mathcal{T}} : [0, 1] \rightarrow \gamma(\mathcal{S})$  satisfying

$$\varphi_{\mathcal{S}\mathcal{T}} \circ T_i(t) = S_i \circ \varphi_{\mathcal{S}\mathcal{T}}(t) \text{ for any } i = 1, \dots, 2m \text{ and } t \in [0, 1]$$

This map is Hölder continuous. We prove the following theorem which allows to find its Hölder exponent:

**Theorem 7.** *Let  $\mathcal{S} = \{S_1, \dots, S_m\}$  be a zipper in  $\mathbb{R}^n$  and suppose all  $S_i$  are similarities.*

*Let  $\varphi : I[0, 1] \rightarrow \gamma(\mathcal{S})$  be the linear parametrization of  $\mathcal{S}$  by a zipper  $\mathcal{T} = \{T_1, \dots, T_m\}$ . The Hölder exponent  $\alpha$  of the map  $\varphi$  satisfies*

$$\alpha = \min_{i=1, \dots, m} \frac{\log \text{Lip } S_i}{\log \text{Lip } T_i}$$

#### **Proof**

Denote  $q_i = \text{Lip } S_i$ ,  $p_i = \text{Lip } T_i$ ,  $p_{\min} = \min_{i=1, \dots, m} p_i$ . Let  $M$  be the diameter of  $\gamma(\mathcal{S})$ .

Observe that for any multiindex  $\mathbf{i} = i_1 \dots i_k$ ,  $q_{\mathbf{i}} \leq p_{i_1}^\alpha$ .

Suppose  $a, b \in I$ ,  $a < b$ . There are two possibilities:

1) For some multiindex  $i_1 \dots i_k i_{k+1}$ ,

$$T_{i_1 \dots i_k i_{k+1}}(I) \subset [a, b] \subset T_{i_1 \dots i_k}(I)$$

In this case,  $p_{i_1 \dots i_k} p_{\min} < |a - b| \leq p_{i_1 \dots i_k}$ , while  $\|f(a) - f(b)\| \leq q_{i_1 \dots i_k} M$ .

But  $q_{i_1 \dots i_k} \leq p_{i_1 \dots i_k}^\alpha$ , so

$$\|f(a) - f(b)\| \leq \frac{M}{p_{\min}^\alpha} |a - b|^\alpha$$

2) For some pair of multiindices  $i_1 \dots i_k i_{k+1}$  and  $j_1 \dots j_l j_{l+1}$ ,

$$T_{i_1 \dots i_k i_{k+1}}(I) \cup T_{j_1 \dots j_l j_{l+1}}(I) \subset [a, b] \subset T_{i_1 \dots i_k}(I) \cup T_{j_1 \dots j_l}(I)$$

In this case,  $(p_{i_1 \dots i_k} + p_{j_1 \dots j_l}) p_{\min} \leq |a - b| \leq (p_{i_1 \dots i_k} + p_{j_1 \dots j_l})$ , while  $\|f(a) - f(b)\| \leq (q_{i_1 \dots i_k} + q_{j_1 \dots j_l}) M$ .

Suppose  $p_{i_1 \dots i_k} \geq p_{j_1 \dots j_l}$ , then  $p_{i_1 \dots i_k} p_{\min} \leq |a - b| \leq 2p_{i_1 \dots i_k}$

Thus, we have

$$\|f(a) - f(b)\| \leq 2M p_{i_1 \dots i_k}^\alpha \leq \frac{2M}{p_{\min}^\alpha} |b - a|^\alpha$$

The exponent  $\alpha$  is exact because for some  $k$ ,  $q_k = p_k^\alpha$ , and therefore  $\|f(a) - f(b)\| = L|a - b|^\alpha$  for  $a = T^n(0)$ ,  $b = T^n(1)$ ,  $L = \|z_0 - z_m\|$ . ■

**Lemma 8.** *There exist such linear zipper  $\mathcal{T}$  on  $[0, 1]$ , that for any  $\xi \in D$ , the linear parametrisation  $\varphi^\xi = \varphi_{\mathcal{S}^\xi \mathcal{T}}$  has Hölder exponent greater than  $3/4$ .*

**Proof** Take some  $\xi_0 = (1, 0, \theta_0) \in D$ . Let  $q_i = \text{Lip } S_i^{\xi_0}$  and  $s$  be the similarity dimension of  $\mathcal{S}^{\xi_0}$ . Take a zipper  $\mathcal{T} = \{T_1, \dots, T_{2m}\}$  on  $[0, 1]$  with signature  $\varepsilon$  and contraction ratios  $p_i = q_i^s$ . Obviously  $\frac{\log q_i}{\log p_i} \equiv 1/s$ . There are only two indices  $i = m + 1$  and  $i = m + 2$ , for which  $\text{Lip } S_i^\xi = q_i'$  depends on  $\xi$ . For both of them

$$|\log q_i' / q_i| \leq \log(1.02), \text{ therefore } 1/1.012 < \frac{\log q_i'}{\log q_i} < 1.012.$$

$$\text{Therefore } \frac{\log p_i}{\log q_i'} < 1.28 \cdot 1.012 < 4/3.$$

It follows that for any  $\xi \in D$ , the linear parametrization  $\varphi^\xi$  of  $\mathcal{S}^\xi$  by the zipper  $\mathcal{T}$  has Hölder exponent greater than  $3/4$ . ■

### 3.2 Verification of A5

First, we reformulate the first statement of Theorem 4 in the following way:

**Lemma 9.** *Let  $\eta_1, \eta_2$  be the generators of a dense subgroup of second type, then for any  $z_1, z_2 \in \mathbb{C} \setminus \{0\}$  and any rays  $l_1, l_2$  issuing from zero, there is such sequence  $\{(n_k, m_k)\}$  that  $\lim_{k \rightarrow \infty} \left| \frac{z_1 \eta_1^{n_k}}{z_2 \eta_2^{m_k}} \right| = 1$ , while the angles between  $l_1, l_2$  and rays passing from 0 to  $z_1 \eta_1^{n_k}$  and  $z_2 \eta_2^{m_k}$  respectively, converge to 0. ■*

Second, the statement of the Lemma remains valid if we replace:

- 1) the plane  $\mathbb{C}$  by a cone  $C \subset \mathbb{R}^3$  with the axis  $L$ ;
- 2) the products  $z_1 \eta_1^{n_k}$  and  $z_2 \eta_2^{m_k}$  by  $f_1^{n_k}(z_1)$  and  $f_2^{m_k}(z_2)$ , where  $f_i$  are compositions of homothety with contraction ratio  $|\eta_i|$  and rotation around the axis  $L$  axis in the angle  $\arg(\eta_i)$  and
- 3) Taking for  $l_1, l_2$  some generators of the cone  $C$ .

Since for any two sequences  $f_k, g_k$  of orthogonal transformations in  $\mathbb{R}^n$  (and therefore for similarities having origin as a fixed point), the convergence of the sequence  $f_k^{-1} g_k$  to Id is equivalent to the convergence of  $g_k f_k^{-1}$  to Id, we have the following

**Lemma 10.** *Let  $PQ$  be a segment in  $\mathbb{R}^3$  and  $\mathbf{n}$  be some its normal vector. Let  $D_k, D'_k$  be two sequences of points in  $\mathbb{R}^3 \setminus \{0\}$  and  $\mathbf{n}_k, \mathbf{n}'_k$  sequences of unit normal vectors to the segments  $OD_k, OD'_k$ . Let  $f_k$  (resp.  $g_k$ ) be the similarities which map  $P$  to  $O$ ,  $PQ$  to  $OD_k$  (resp.  $OD'_k$ ) and whose orthogonal parts send  $\mathbf{n}$  to  $\mathbf{n}_k$  (resp.  $\mathbf{n}'_k$ ). Then*

$$f_k^{-1} g_k \rightarrow \text{Id} \quad \text{iff} \quad \frac{|OD_k|}{|OD'_k|} \rightarrow 1, (\mathbf{n}_k, \mathbf{n}'_k) \rightarrow 1, \angle D_k O D'_k \rightarrow 0. \quad \blacksquare$$

**Lemma 11.** *For any  $\xi \in D$ , there is such sequence  $\Sigma_\xi = \{(i_k, j_k)\}$  that the similarities  $S'_k = S_m S_{2m}^{i_k}$  and  $S''_k = S_{m+1} S_1^{j_k}$  satisfy*

$$\lim_{k \rightarrow \infty} (S'_k S_{m+4})^{-1} (S''_k S_{m-3}) = \text{Id}.$$

**Proof** Observe that  $S_{m+4}^{-1}(z_{2m}) = S_{m-3}^{-1}(z_0) = (-153, 0.8, 0)$ . Denote this point by  $P$ . Let  $Q = (3, 0.8, 0)$ . Each map  $S_m S_{2m}^k S_{m+4}$  sends  $P$  to the point  $z_m = (0, 0, 0)$  and the point  $Q$  to some point on the surface of the cone  $S_m(V^0)$ . Each map  $S_{m+1} S_1^k S_{m-3}$  sends  $P$  to the point  $z_m = (0, 0, 0)$

and the point  $Q$  to some point on the surface of the cone  $S_{m+1}(V^0)$ . The maps  $f_1 = S_m S_{2m} S_m^{-1}$  and  $f_2 = S_{m+1} S_1 S_{m+1}^{-1}$  are the similarities preserving the cones  $S_m(V^0)$  and  $S_{m+1}(V^0)$  respectively, defined by two generators of a dense group of second kind.

Let  $l$  be a common generator of these intersecting cones.

Denoting  $w_1 = S_m \circ S_{m+4}(Q)$  and  $w_2 = S_{m+1} \circ S_{m-3}(Q)$ , rewrite

$$S_m S_{2m}^k S_{m+4}(Q) = f_1^k(S_m \circ S_{m+4}(Q)) = f_1^k(w_1)$$

and

$$S_{m+1} S_1^k S_{m-3}(Q) = f_2^k(S_{m+1} \circ S_{m-3}(Q)) = f_2^k(w_2)$$

Since  $w_1, w_2$  lie on the surfaces of the cones  $S_m(V^0)$  and  $S_{m+1}(V^0)$ , we may apply Lemma 9 to get subsequences  $i_k, j_k$  for which  $\lim_{k \rightarrow \infty} \frac{\|f_1^{i_k}(w_1)\|}{\|f_2^{j_k}(w_2)\|} = 1$ , while the angles between  $l$  and rays passing from 0 to  $f_1^{i_k}(w_1)$  and  $f_2^{j_k}(w_2)$  respectively, converge to 0.

At the same time, since  $S_{m+4}$  contains a rotation in the angle  $-\alpha_{m+4}$ , which is the angle between normals to these cones at points of  $l$ , the angle between the images of the normal  $\mathbf{n}$  to the segment  $PQ$  under the maps  $S'_k S_{m+4} = S_m S_{2m}^{i_k} S_{m+4}$  and  $S''_k S_{m-3} = S_{m+1} S_1^{j_k} S_{m-3}$  converges to 0. Therefore, by Lemma 10,

$$\lim_{k \rightarrow \infty} (S'_k S_{m+4})^{-1} (S''_k S_{m-3}) = \text{Id.} \quad \blacksquare$$

### 3.3 Dividing $S_m(\gamma) \cap S_{m+1}(\gamma)$ to a sequence of disjoint pieces: checking (A4).

**Lemma 12.** *There is such sequence  $\Sigma = \{(i_k, j_k)\}$  in  $\mathbb{N} \times \mathbb{N}$  that:*

(1) *For any  $\xi \in D$ ,*

$$S_m(\gamma) \cap S_{m+1}(\gamma) = \{z_m\} \cup \left( \bigcup_{i,j=1}^{\infty} (S_{m+1} S_1^{i_k}(\gamma_A) \cap S_m S_{2m}^{j_k}(\gamma_B)) \right)$$

(2) *For any  $k$  there is such  $\xi \in D$  that*

$$S_{m+1} S_1^{i_k}(A) \cap S_m S_{2m}^{j_k}(B) \neq \emptyset$$

(3) *If a pair  $(i, j) \notin \Sigma$ , then for any  $\xi \in D$ ,*

$$S_{m+1} S_1^i(A) \cap S_m S_{2m}^j(B) = \emptyset$$

- (4) The sequences  $\{(i_k)\}$  and  $\{(j_k)\}$  are strictly increasing and both projections  $Pr_1 : (i, j) \rightarrow i$ ,  $Pr_2 : (i, j) \rightarrow j$  are injective on  $\Sigma$ ,  
(5) For any  $\xi \in D$ ,  $\Sigma_\xi \subset \Sigma$ .

**Proof**

For any system  $\mathcal{S}_\xi, \xi \in D$ , its invariant set  $\gamma^\xi$  is a Jordan arc if  $S_m(\gamma) \cap S_{m+1}(\gamma) = \{z_m\}$ .

Consider the set  $S_m(\gamma) \cap S_{m+1}(\gamma)$ . It is contained in  $S_m(V) \cap S_{m+1}(V)$ . By Lemma 5,  $S_m(V^1) \cap S_{m+1}(V) = S_m(V) \cap S_{m+1}(V^1) = \{z_m\}$ . Therefore,

$$S_m(V) \cap S_{m+1}(V) = S_m(V \setminus \dot{V}^1) \cap S_{m+1}(V \setminus \dot{V}^1)$$

The intersection of  $\gamma \setminus \{z_0, z_{2m}\}$  and  $V \setminus \dot{V}^1$  lies in the set

$$\left( \bigcup_{n=0}^{\infty} S_1^n(A) \right) \cup \left( \bigcup_{n=0}^{\infty} S_{2m}^n(B) \right)$$

therefore

$$S_m(\gamma) \cap S_{m+1}(\gamma) \subset \{z_m\} \cup \left( \bigcup_{i,j=1}^{\infty} (S_{m+1}S_1^i(A) \cap S_mS_{2m}^j(B)) \right)$$

Suppose  $x \in A$ ,  $y \in B$  and  $S_{m+1}S_1^i(x) = S_mS_{2m}^j(y)$ . Denote  $a = \log(\|x - z_0\|)$ ,  $b = \log(\|y - z_{2m}\|)$ . Then  $\log(q_{m+1}) + i \log(q_1) + a = \log(q_m) + j \log(q_{2m}) + b$ , or

$$|i \log(q_1) - j \log(q_{2m})| \leq |b - a| + \log(q_{m+1}/q_m)$$

According to the inequality (1),  $|b - a| < \log 1.06 < 0.06$ . At the same time,  $|\log q_{m+1}/q_m| < 0.04$ . Therefore  $i$  and  $j$  are the solutions of the inequality

$$|i \log(q_1) - j \log(q_{2m})| < 0.1 \quad (**)$$

Both  $q_1$  and  $q_{2m}$  lie between  $1/5$  and  $1/7$ , so absolute values of their logarithms are greater than 1.

This implies that for any  $i \in \mathbb{N}$  there is at most one  $j \in \mathbb{N}$  for which the inequality (\*\*) holds and vice versa. The solutions of this inequality are the same for all values of  $q_{m+1}$  satisfying  $0.98 < \frac{q_{m+1}}{q_m} < 1.02$ .

Therefore there is a subsequence  $\{(i_k, j_k)\}$  of  $\Sigma$  which runs through all non-negative solutions of the inequality (\*\*) for any value of  $q_{m+1}$ ; both



sequences  $i_k$  and  $j_k$  being strictly increasing. Obviously, it will contain  $\Sigma_\xi$  for any  $\xi \in D$ . Discarding those entries  $\{(i_k, j_k)\}$ , for which  $S_{m+1}S_1^{i_k}(A) \cap S_mS_{2m}^{j_k}(B) = \emptyset$  for any  $\xi \in D$ , we get the desired sequence.

Thus, for any value of  $q_{m+1}$ , the set  $S_m(\gamma) \cap S_{m+1}(\gamma) \setminus \{z_m\}$  is the disjoint union  $\bigcup_{i=0}^{\infty} (S_{m+1}S_1^{i_k}(\gamma_A) \cap S_mS_{2m}^{j_k}(\gamma_B))$  ■

## 4 Proof of the main Theorem

To prove Theorem 1 we make the following steps:

First we make the estimates for the difference  $\|S_i^\xi(x) - S_i^\eta(x)\|$  on the set  $V$  for any given  $\xi, \eta \in D$  for  $i = 1, \dots, 2m$ . They are quite simple for  $i \neq m+1$  (Lemma 13), so most of the work is done for  $i = m+1$ . For each  $k \in \mathbb{N}$  we express these estimates in terms of the displacement  $\delta_k^* = x_k^\xi - x_k^\eta$  of the point  $x_k = S'_k(z_{m-4})$ . We prove in Lemma 17 that the dependence of  $x_k^\xi$  on  $\xi$  is bi-Lipschitz. In Lemma 18 we estimate  $\max_{i=1, \dots, 2m, x \in V} \|S_i^\xi(x) - S_i^\eta(x)\|$  in terms of  $\delta_k^*$ . In Lemma 19 we get upper and lower bounds for  $\|S_{m+1}^\xi(x) - S_{m+1}^\eta(x)\|$  on  $S_1^{i_k}(A)$  in terms of  $\delta_k^*$ . In the Subsection 4.3 we prove Theorem 2 and Proposition 3; in Lemma 20 we prove that the function  $f(\xi, s, t)$  used in the Theorem 2 is bi-Lipschitz with respect to  $\xi$  and  $\alpha$ -Hölder with respect to  $s, t$  which gives us the proof of the main Theorem.

**Some notation.** Let  $\xi = (\rho_1, \varphi_1, \theta_1)$ ,  $\eta = (\rho_2, \varphi_2, \theta_2)$  and  $\Delta\xi = (\Delta\rho, \Delta\varphi, \Delta\theta) = \xi - \eta$ . Denote by  $x_k^\xi$  the point  $S'_k(z_{m-4}) = S_{m+1}^\xi S_1^{i_k}(z_{m-4})$  and let  $\delta_k^* = \|x_k^\xi - x_k^\eta\|$ .

Fixing the value of  $\xi$ , we consider spherical coordinate system whose polar axis is  $z_m z_{m+1}^\xi$  and whose azimuth direction is a perpendicular to  $z_m z_{m+1}^\xi$  lying in the right half-plane of the plane  $XY$ . We'll denote these coordinates by  $\varrho, \phi, \vartheta$  and call them  $\xi$ -coordinates.

It is more convenient to represent the difference  $\|S_i^\xi(x) - S_i^\eta(x)\|$ ,  $x \in A$  as  $\|S_{m+1}^\eta (S_{m+1}^\xi)^{-1}(x) - x\|$ , where  $x \in S_{m+1}^\xi S_1^{i_k}(A)$ . Denote  $F_{\xi\eta} = S_{m+1}^\eta (S_{m+1}^\xi)^{-1}$ .

Each of the maps  $F_{\xi\eta}$  may be viewed as a composition of a rotation  $R_\varphi$  in an angle  $\Delta\varphi$  with respect to the line  $z_m z_{m+1}^\xi$ , a rotation  $R_\theta$  in an angle  $\Delta\theta$  with respect to  $Z$  axis and homothety  $H_\rho$  with ratio  $\rho_2/\rho_1$ .

## 4.1 Transition maps from $\mathcal{S}^\xi$ to $\mathcal{S}^\eta$ and their estimates.

First we consider all  $i \neq m+1$ :

**Lemma 13.** *For any  $x \in V$ ,  $\xi, \eta \in D$ ,*

- (a) *If  $i \neq m+1, m+2$  or  $m+4$ , then  $S_i^\xi(x) - S_i^\eta(x) \equiv 0$ ;*
- (b)  *$\|S_{m+4}^\xi(x) - S_{m+4}^\eta(x)\| < 0.46|\Delta\theta|$ ;*
- (c)  *$\|S_{m+2}^\xi(x) - S_{m+2}^\eta(x)\| \leq \|z_{m+1}^\xi - z_{m+1}^\eta\|$*

**Proof.** For  $i = m+4$ ,

$$\|S_{m+4}^\xi(x) - S_{m+4}^\eta(x)\| \leq q_{m+4}|\alpha_{m+4}^\xi - \alpha_{m+4}^\eta|r_x,$$

where  $r_x$  is a distance from the point  $x$  to the OX axis. Therefore,

$$\|S_{m+4}^\xi(x) - S_{m+4}^\eta(x)\| < 20q_{m+4}|\Delta\theta|r_x < 0.298|\Delta\theta|r_x$$

Thus,  $r_x \leq 1.538$  implies  $\max_{x \in V} \|S_{m+4}^\xi(x) - S_{m+4}^\eta(x)\| < 0.46|\Delta\theta|$ .

The similarity  $S_{m+2}^\xi$  depends only on a position of the point  $z_{m+1}$ , therefore

$$\|S_{m+2}^\xi(x) - S_{m+2}^\eta(x)\| \leq \frac{\|x - z_{2m}\|}{\|z_0 - z_{2m}\|} \|z_{m+1}^\xi - z_{m+1}^\eta\| \quad \blacksquare$$

The estimates for  $i = m+1$  are more complicated.

The following two lemmas will help us to estimate maximal and minimal displacement for different points in  $V_{m+1}$ :

**Lemma 14.** *Let  $S$  be a similarity in  $\mathbb{R}^3$  which is a composition of a homothety with a fixed point  $c$  and a rotation around line  $l \ni c$  and let  $P$  be a plane containing line  $l$ . Let  $B(a, r)$  be a closed ball disjoint from  $P$ . Then*

$$\max_{x, y \in B} \frac{\|S(x) - x\|}{\|S(y) - y\|} \leq \frac{d(a, P) + r}{d(a, P) - r} \quad \blacksquare$$

**Lemma 15.** *Suppose  $0 < \alpha < \pi/2$ , and  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are such unit vectors that the angles between each  $\mathbf{e}_i, \mathbf{e}_j$  lie between  $\pi/2 - \alpha$  and  $\pi/2 + \alpha$ . Let  $\lambda_1 < r_i < \lambda_2$ . Then, for any  $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}$ ,*

$$\lambda_1(\sqrt{1 - 2\sin\alpha})\|\mathbf{x}\| \leq \left\| \sum_{i=1}^3 x_i r_i \mathbf{e}_i \right\| \leq \lambda_2(\sqrt{1 + 2\sin\alpha})\|\mathbf{x}\|. \quad \blacksquare$$

Using Lemma 12 and Lemma 5, we find possible  $\xi$ -coordinates of the point  $x_k^\xi$ :

**Lemma 16.** *If  $S'_k(A) \cap S''_k(B) \neq \emptyset$ , then the point  $x_k^\xi$  has  $\xi$ -coordinates  $(r_k, \phi, \beta_0)$ , where  $r_k = \sqrt{5} \text{Lip}(S'_k)$ ,  $-\alpha_c < \phi < \alpha_c$ , and*

$$\alpha_c = \arccos \frac{\tan(\beta - \mu/2)}{\tan(\beta + \mu)} + \sqrt{5}\mu = 0.295 \quad \blacksquare$$

Using last three Lemmas we show that the map, which assigns to each  $\xi \in D$  the point  $x_{k+1}^\xi$ , is bi-Lipschitz:

**Lemma 17.** *For any  $k$ , the map  $g_k(\xi) = S_{m+1}^\xi S_1^{i_k}(z_{m-4})$  is bi-Lipschitz with respect to  $\xi$  on  $D$ .*

**Proof.** Since  $\xi, \eta \in D$ ,  $|\Delta\rho| < 0.02\rho$ ,  $|\Delta\varphi| < 2\mu$ ,  $|\Delta\theta| < \mu/2$ , the distances from  $x_k$  to the closest fixed point of the maps  $H_\rho, R_\varphi$  and  $R_\theta$  respectively are  $r_k$ ,  $r_k \sin \beta_0 = \frac{r_k}{\sqrt{5}}$  and  $r_k \sqrt{\cos^2 \beta_0 + \sin^2 \beta_0 \cos^2 \varphi}$ . Each of these values lies between  $r_k/\sqrt{5}$  and  $r_k$ .

The displacement vectors corresponding to each of these maps, have norms  $\frac{r_k \Delta\rho}{\rho}$ ,  $\frac{2r_k}{\sqrt{5}} \sin \frac{\Delta\varphi}{2}$  and  $2r_k \sqrt{\frac{5 - 4 \sin^2 \varphi}{5}} \cdot \sin^2 \frac{\Delta\theta}{2}$ , whose values, divided by  $r_k \Delta\rho$ ,  $r_k \Delta\varphi$  and  $r_k \Delta\theta$  respectively lie in the interval  $\left(\frac{0.98}{\sqrt{5}}, 1\right)$ .

The angles between each two of these vectors belong to  $\left(\frac{\pi}{2} - \alpha_c, \frac{\pi}{2} + \alpha_c\right)$ . Applying Lemma 15 and taking into account, that  $\sin \alpha_c < 0.295$ , we have

$$\frac{0.98r_k}{\sqrt{5}} \sqrt{0.41} \|\Delta\xi\| < \|x_k^\eta - x_k^\xi\| < r_k \sqrt{1.59} \|\Delta\xi\| \quad \blacksquare$$

From the last inequality in the proof we obtain

$$1.26 \frac{\delta_k^*}{r_k} < \sqrt{\Delta\rho^2 + \Delta\varphi^2 + \Delta\theta^2} < 3.56 \frac{\delta_k^*}{r_k}$$

The maximal displacement in the subset  $V_{m+1}$  is reached at the point  $z_{m+1}$  and is less or equal to  $\rho \sqrt{\Delta\rho^2 + \Delta\theta^2} < 3.56 \frac{\delta_k^*}{r_k} \rho < 3.64 \frac{\delta_k^*}{r_k}$ .

Since  $\Delta\theta < 3.56 \frac{\delta_k^*}{r_k}$ , by Lemma 13 for any  $x \in V$ ,  $\|S_{m+4}^\xi(x) - S_{m+4}^\eta(x)\| < 1.64 \frac{\delta_k^*}{r_k}$ , so we have

**Lemma 18.** For any  $x \in V$  and any  $i = 1, \dots, 2m$ ,

$$\|S_i^\eta(x) - S_i^\xi(x)\| < 3.64 \frac{\delta_k^*}{r_k} \quad \blacksquare$$

Denote this upper bound for the displacement,  $3.64 \frac{\delta_k^*}{r_k}$  by  $\delta_k$ .

## 4.2 Estimates for $F_{\xi\eta}$

Now we take  $(i_k, j_k) \in \Sigma$ ,  $\xi, \eta \in D$  and estimate the distances between the points of  $S_{m+1}^\xi S_1^{i_k}(\gamma_A^\xi)$  and  $S_{m+1}^\eta S_1^{i_k}(\gamma_A^\eta)$  having the same addresses.

To apply the Proposition 3 we first prove the following

**Lemma 19.** For any  $x \in S_1^{i_k}(A)$ ,  $\delta_k^*/1.19 < \|S_{m+1}^\xi(x) - S_{m+1}^\eta(x)\| < 1.19\delta_k^*$

### Proof

The map  $F_{\xi\eta}$  is a composition of a homothety with ratio  $\rho_2/\rho_1$  and a rotation  $R_{\xi\eta}$ . The map  $F_{\xi\eta}$  sends the point with spherical coordinates  $(\rho_1, 0, 0)$  to the point  $(\rho_2, 0, \Delta\theta)$ , so  $R_{\xi\eta}$  sends  $(1, 0, 0)$  to  $(1, 0, \Delta\theta)$ . Therefore the axis  $l$  of the rotation  $R_{\xi\eta}$  lies in the middle bisector plane for these points and the unit normal vector to this plane has the coordinates  $\left(1, 0, \frac{\pi + \Delta\theta}{2}\right)$ .

Therefore the set  $S_{m+1}^\xi S_1^{i_k}(A)$  can be covered by a ball  $W_k$  whose radius is  $0.036R_k$  (where  $R_k = q_{m+1}^\xi q_1^{i_k} R$ ) and whose center has a coordinate  $(1.03R_k, \phi, \beta_0 - \mu/2)$ . It follows from Lemma 16 that  $-\alpha_c \leq \phi \leq \alpha_c$ .

The maximal angle between the coordinate polar axis  $l_\xi$  and the plane  $P_l$ , containing the axis  $l$  of  $F_{\xi\eta}$  is  $\mu/4$ , therefore minimal possible distance between the center of  $W_k$  and the plane  $P_l$  is greater than  $0.43R_k$ .

It follows from the Lemma 14 that  $\max_{x, y \in W_k} \frac{\|F_{\xi\eta}(x) - x\|}{\|F_{\xi\eta}(y) - y\|} \leq 1.19$ .

Taking  $x = x_k$  and  $\|F_{\xi\eta}(x_k) - x_k\| = \delta_k^*$ , for any other point  $y \in S_{m+1}^\xi S_1^{i_k}(A)$ , we have

$$\delta_k^*/1.19 < \|F_{\xi\eta}(y) - y\| < 1.19\delta_k^* \quad \blacksquare$$

### 4.3 Proof of Theorems 2 and Proposition3; proof of Theorem1.

#### Proof of the Theorem 2.

Since functions  $\varphi(x, t)$  and  $\psi(x, t)$  are  $\alpha$ -Hölder with respect to  $t$ , then  $f(x, s, t)$  is also  $\alpha$ -Hölder with respect to  $(s, t)$ .

Since  $f(x, s, t)$  is biLipschitz with respect to  $x$ , it is clear that if  $f(x_1, s, t) = f(x_2, s, t)$ , then  $x_1 = x_2$ , so the equation  $f(x, s, t) = 0$  specifies an implicit function  $x = g(s, t)$  defined on some closed subset  $P \subset [0, 1] \times [0, 1]$ .

Take  $x_1 = g(s_1, t_1)$  and  $x_2 = g(s_2, t_2)$  in  $g(P)$ . Since  $f$  is bi-Lipschitz w.r.t.  $x$ ,  $\|f(x_2, s_1, t_1) - f(x_1, s_1, t_1)\| \geq \|x_1 - x_2\|/L$ . Since  $f$  is  $\alpha$ -Hölder w.r.t.  $(s, t)$ ,  $\|f(x_2, s_1, t_1) - f(x_2, s_2, t_2)\| \leq M\|(s_1 - s_2, t_1 - t_2)\|^\alpha$ .

Therefore  $|x_1 - x_2| \leq LM \|(s_1 - s_2, t_1 - t_2)\|^\alpha$ .

Thus, the function  $g(s, t)$  is  $\alpha$ -Hölder on the set  $P$ .

From  $\dim_H \leq 2$  we obtain  $\dim_H(g(P)) \leq \frac{2}{\alpha}$ . ■

**Proof of the Proposition 3.** For  $\sigma = i_1 i_2 i_3 \dots$  denote  $\sigma_k = i_{k+1} i_{k+2} \dots$ . For  $\mathbf{j} = j_1 \dots j_k$ ,  $\hat{\sigma}_{\mathbf{j}}$  is an operator sending  $i_1 i_2 i_3 \dots$  to  $j_1 \dots j_k i_1 i_2 i_3 \dots$ .

By Barnsley Collage Theorem,  $|\psi(\sigma) - \varphi(\sigma)| \leq \frac{\delta}{1 - q}$ .

Let  $\mathbf{i} = i_1 \dots i_k$ ,  $\mathbf{j} = j_1 \dots j_l$ .

Write  $\psi(\sigma) - \varphi(\sigma) = (T_{\mathbf{i}}\psi(\sigma_k) - S_{\mathbf{i}}\psi(\sigma_k)) + (S_{\mathbf{i}}\psi(\sigma_k) - S_{\mathbf{i}}\varphi(\sigma_k))$ . We have  $\delta_1 \leq \|T_{\mathbf{i}}\psi(\sigma_k) - S_{\mathbf{i}}\psi(\sigma_k)\| \leq \delta_2$  and  $\|S_{\mathbf{i}}\psi(\sigma_k) - S_{\mathbf{i}}\varphi(\sigma_k)\| \leq \frac{q_i \delta}{1 - q}$ , which implies **B1**

The same way  $\psi(\sigma) - \varphi(\sigma) - \psi(\tau) + \varphi(\tau) = [(T_{\mathbf{i}}\psi(\sigma_k) - S_{\mathbf{i}}\psi(\sigma_k)) - (T_{\mathbf{j}}\psi(\tau_k) - S_{\mathbf{j}}\psi(\tau_k))] + [(S_{\mathbf{i}}\psi(\sigma_k) - S_{\mathbf{i}}\varphi(\sigma_k)) - (S_{\mathbf{j}}\psi(\tau_k) - S_{\mathbf{j}}\varphi(\tau_k))]$ .

The norm of first brackets lies between  $\delta_1$  and  $\delta_2$ , the norm of second brackets is no greater than  $\frac{(q_i + q_j)\delta}{1 - q}$ , which gives us **B2**. ■

Let  $\varphi^\xi : [0, 1] \rightarrow \gamma^\xi$  be the linear parametrization of the zipper  $\mathcal{S}_\xi$  defined in Lemma 8. Let  $I_{Ak} = T_{m+1}T_1^{i_k}(I_A)$  and  $I_{Bk} = T_mT_{2m}^{j_k}(I_B)$  be the subintervals of  $I = [0, 1]$ , for which  $\varphi^\xi(I_{Ak}) = S_{m+1}S_1^{i_k}(\gamma_A)$  and  $\varphi^\xi(I_{Bk}) = S_mS_{2m}^{j_k}(\gamma_B)$ . Denote  $\varphi^\xi|_{I_{Ak}}$  by  $\varphi(\xi, t)$  and  $\varphi^\xi|_{I_{Bk}}$  by  $\psi(\xi, t)$ . Take  $s \in I_{Ak}$ ,  $t \in I_{Bk}$ .

**Lemma 20.** *The function  $f(\xi, s, t) = \varphi(\xi, s) - \psi(\xi, t)$  is bi-Lipschitz with respect to  $\xi$  for any  $s \in I_{Ak}$ ,  $t \in I_{Bk}$ .*

**Proof**

Observe that if  $S_{m+1}S_1^{i_k}(A) \cap S_mS_{2m}^{j_k}(B) \neq \emptyset$ , then  $\frac{q_m q_{2m}^{j_k}}{q_{m+1} q_1^{i_k}} < 1.06$ .

We apply the Proposition 3 to  $S_{m+1}S_1^{i_k}(A)$  and  $S_mS_{2m}^{j_k}(B)$ .

Notice that  $q_{m+4}$  is greater than  $q_{m+3}$  and  $q_{m+5}$  and the same is true for their symmetric counterparts, so we use  $q_{m+4}$  (which is equal to  $q_{m-4}$ ) for both  $S_{m+1}S_1^{i_k}(A)$  and  $S_mS_{2m}^{j_k}(B)$ .

By Proposition 3, for any  $s \in I_{Ak}$ ,  $t \in I_{Bk}$ ,

$$\frac{\delta_k^*}{1.19} - \frac{q_{m+4}\delta_k}{1-q}(q_{m+1}q_1^{i_k} + q_m q_{2m}^{j_k}) \leq |\varphi(\xi, s) - \varphi(\eta, s) - \psi(\xi, t) + \psi(\eta, t)| \leq$$

$$1.19\delta_k^* + \frac{q_{m+4}\delta_k}{1-q}(q_{m+1}q_1^{i_k} + q_m q_{2m}^{j_k})$$

Evaluating  $\frac{q_{m+4}}{1-q}(q_{m+1}q_1^{i_k} + q_m q_{2m}^{j_k})\delta_k < 0.02 \cdot 2.06q_{m+1} \cdot 3.64\delta^* < 0.03\delta_k^*$ , we get

$$0.8\delta_k^* < |\varphi(\xi, s) - \varphi(\eta, s) - \psi(\xi, t) + \psi(\eta, t)| < 1.22\delta_k^*$$

Since for the point  $x_k$ , the function  $\varphi(\xi, \eta) = F_{\xi\eta}(x_k) - x_k$  is bi-Lipschitz with respect to  $\xi$  and  $\eta$ , and since  $\|F_{\xi\eta}(x_k) - x\| = \delta_k^*$ , the last inequality shows that the function  $f(\xi, s, t) = \varphi(\xi, s) - \psi(\xi, t)$  is bi-Lipschitz with respect to  $\xi$ . ■

## 5 Addendum: Dense groups of second type and their generators.

We will remind several facts about dense 2-generator subgroups in  $\mathbb{C}^*$ . See [3] for details.

As it follows from Kronecker's Theorem

**Proposition 21.** *Let  $u, v \in \mathbb{C}$ ,  $Im \frac{u}{v} \neq 0$ ,  $\alpha u + \beta v = 1$ ,  $\alpha, \beta \in \mathbb{R}$ ,  $\xi = e^{2\pi i u}$ ,  $\eta = e^{2\pi i v}$ .*

*A group  $G = \langle \xi, \eta, \cdot \rangle$  is dense in  $\mathbb{C}$  iff for any integers  $k, l, m$ ,*

$$k\alpha + l\beta + m = 0 \quad \text{implies} \quad k = l = m = 0. \quad \blacksquare$$

Then  $G = \langle \xi, \eta, \cdot \rangle$  is called a *dense 2-generator multiplicative group* in  $\mathbb{C}$ .

Notice that if  $G = \langle \xi, \eta, \cdot \rangle$  is dense in  $\mathbb{C}$ , the formula  $\psi(\xi^m \eta^n) = \left( \frac{\xi}{|\xi|} \right)^m$  defines a homomorphism  $\psi$  of a group  $G = \langle \xi, \eta, \cdot \rangle$  to the unit circle  $S^1 \subset \mathbb{C}$ . Put

$$H_G = \bigcap_{\varepsilon > 0} \overline{\psi(B(1, \varepsilon) \cap G)}$$

In other words,  $H_G$  is the set of limit points of all those sequences  $\{e^{in_k \arg(\xi)}\}$ , for which  $\{n_k\}$  are the first coordinates of such sequence  $\{(n_k, m_k)\}$ , that  $\lim_{k \rightarrow \infty} \xi^{n_k} \eta^{m_k} = 1$ .

The set  $H_G$  is a closed topological subgroup of the unit circle  $S^1$ , so it is either finite cyclic or infinite.

**Definition 22.** A dense 2-generator subgroup  $G$  is called the group of first type, if  $H_G$  is finite, and the group of second type, if  $H_G = S^1$ .

So  $G$  is of second type iff for some  $\alpha \notin \mathbb{Q}$ ,  $e^{2\pi i \alpha} \in H_G$ .

Therefore, if  $\xi, \eta$  are the generators of a group of second type, then for any rational  $p, q$ , the numbers  $\xi^p, \eta^q$  also generate a group of the second type. This implies

**Proposition 23.** The set of pairs  $\xi, \eta$  of generators of the groups of second type is dense in  $\mathbb{C}^2$ . ■

The groups of second type have a significant geometric property:

**Theorem 24.** If the group  $G = \langle \xi, \eta, \cdot \rangle$ ,  $\xi = re^{i\alpha}, \eta = Re^{i\beta}$  is of second type, then for any  $z_1, z_2 \in \mathbb{C} \setminus \{0\}$  there is such sequence  $\{(n_k, m_k)\}$  that  $\lim_{k \rightarrow \infty} \frac{z_1 \xi^{n_k}}{z_2 \eta^{m_k}} = 1$ ,  $\lim_{k \rightarrow \infty} e^{in_k \alpha} = e^{-i \arg(z_1)}$ ,  $\lim_{k \rightarrow \infty} e^{in_k \beta} = e^{-i \arg(z_2)}$ .

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